Simply Connected Domains

Definition (simply connected domain) A domain D is called simply connected if it has the following property: if C is any simple closed contour lying in D and z is interior to C, then $z \in D$. //

Intuitively, a simply connected domain is a domain that has no "holes."

Examples (simply connected) • Open disks • Complex plane • interior of any simple closed curve.

(not simply connected domains) • deleted open disks • $\mathbb{C} \setminus \{p\}$

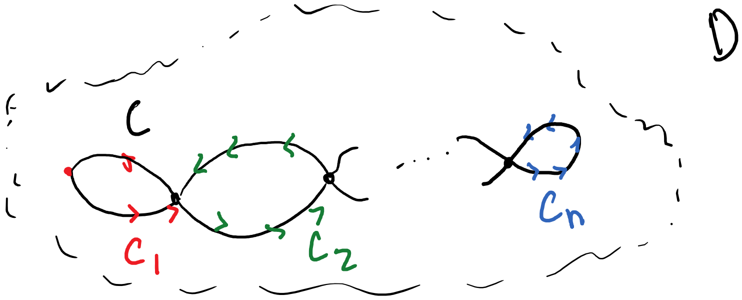
A result similar to the Cauchy-Goursat theorem can be proved for closed contours that may not be simple, provided that they lie in a simply connected domain.

Theorem (Cauchy-Goursat for Simply Connected Domains)
Suppose that f is analytic on a simply connected domain D. If C is any closed contour lying in D, then

$$\int_C f(z)\,dz = 0.$$

Proof. (We will assume this holds for contours with infinitely many self-intersections, without proof)

Assume C is a closed contour w/ finitely many self intersections.



Then C is made up of a finite number of simple closed contours $C_1, \ldots, C_n$. So

$$\int_C f = \sum_{i=1}^{n} \int_{C_i} f = 0$$

by applying Cauchy-Goursat to each integral $\int_{C_i} f$. ∎

$\boxed{\text{Corollary (Antiderivatives of analytic functions)}}$ If f is analytic on a simply connected domain D, then f has an antiderivative on D.

Proof. By the preceding theorem, $\int_C f(z)dz = 0$ for any closed contour lying in D. By the Fundamental Thm of Contour Integrals, this is equivalent to f having an antiderivative on D. ∎

$\boxed{\text{Corollary (Entire functions have antiderivatives)}}$ Suppose that f is entire. Then f has an antiderivative on $\mathbb{C}$.

Proof. The complex plane $\mathbb{C}$ is simply connected.

Apply the preceding corollary. ∎

Multiply Connected Domains

Definition (Multiply Connected) A domain D is called *multiply connected* if it is not simply connected.

The Cauchy-Goursat theorem is easily generalized to multiply connected domains with a finite number of holes.

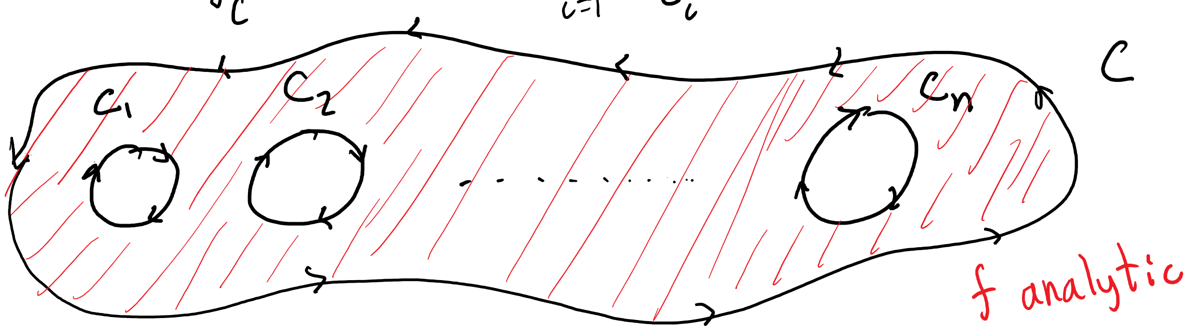
Theorem (Generalized Cauchy-Goursat) Suppose that

(1) C is a simple closed positively oriented contour.

(2) $C_1, \dots, C_n$ are simple closed negatively oriented contours enclosing regions $R_1, \dots, R_n$. Furthur assume that the regions are pairwise disjoint and interior to C.

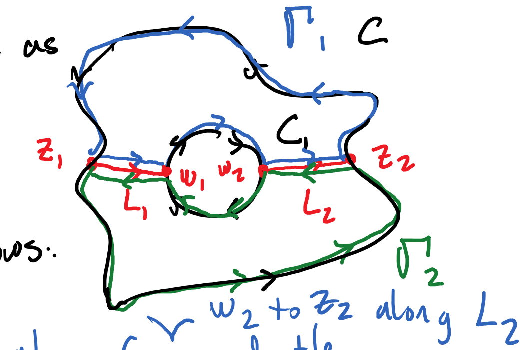
If f is analytic on each contour and the region consisting of all points interior to C but exterior to each C_i, then

$$\int_C f(z)\,dz + \sum_{i=1}^{n} \int_{C_i} f(z)\,dz = 0.$$



Proof. By induction.

Base case: $n=1$. Assume C and C_1 are contours satisfying the hypotheses. Let z_1, z_2, w_1, w_2 be as in the picture. Join z_1 to w_1 with a polygonal line L_1. Also, join w_2 to z_1 with a polygonal line L_2. Define contours as follows:



Γ_1: follow z_1 to w_1 along L_1, then w_1 to w_2 along C_1 and then z_2 to z_1 along C.

Γ_2: follow z_2 to w_2 along $-L_2$ and w_2 to w_1 along C_1 and w_1 to z_1 along $-L_1$ and z_1 to z_2 along C.

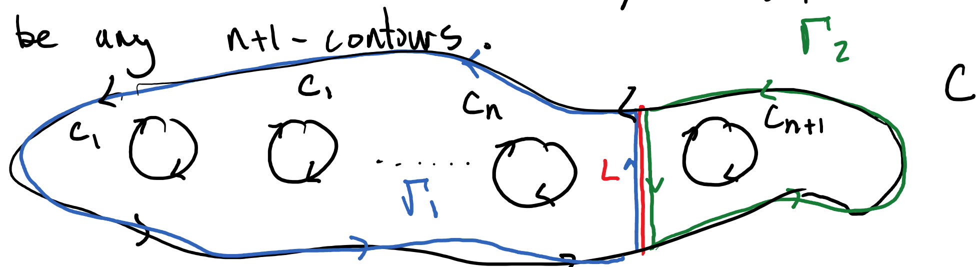
Then f is analytic inside and on the simple closed curves Γ_1 and Γ_2, so by the Cauchy-Goursat theorem,

$$\int_C f(z)\,dz = \int_{\Gamma_1} f(z)\,dz + \int_{\Gamma_2} f(z)\,dz$$

$$\overset{\text{Cauchy-Goursat}}{=} 0 + 0$$

$$= 0.$$

Inductive step: assume $n \geq 1$ and $\int_C f + \sum \int_{C_i} f = 0$ for any n curves $C_1, \ldots, C_n$ satisfying the hypotheses. Let $C_1, \ldots, C_n, C_{n+1}$ be any $n+1$-contours.



Introduce a polygonal line L that seperates $C_1, \ldots, C_n$ from C_{n+1}. Let Γ_1 and Γ_2 be curves defined as in the picture. Then

$$\int_C f = \int_{\Gamma_1} f + \int_{\Gamma_2} f.$$

By inductive hypothesis $\int_{\Gamma_1} f = -\sum_{i=1}^{n} \int_{C_i} f$. By the case $n=1$, $\int_{\Gamma_2} f = -\int_{C_{n+1}} f$. Hence,

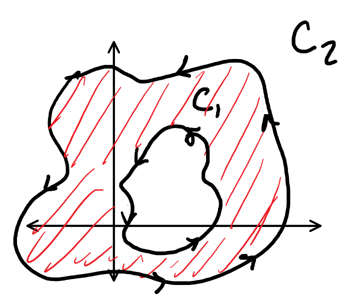
$$\int_C f = -\sum_{i=1}^{n} \int_{C_i} f + \left(-\int_{C_{n+1}} f \right) = -\sum_{i=1}^{n+1} \int_{C_i} f.$$

$\blacksquare$

$\boxed{\text{Corollary (Principle of Deformation of Paths)}}$

Suppose that C_1 and C_2 are positively oriented simple closed contours with C_1 interior to C_2. If f is analytic on the region consisting of C_1, C_2 and all the points between them, then

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz.$$



Proof. Apply the Generalized Cauchy-Goursat theorem to C_2 and $-C_1$ to get

$$\int_{C_2} f + \int_{-C_1} f = 0.$$

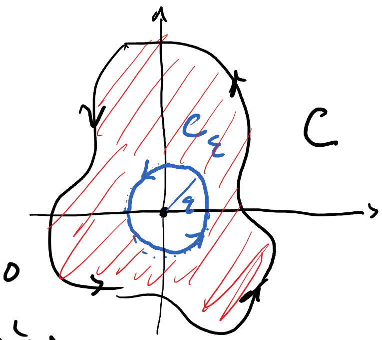
Since $\int_{-C_1} f = -\int_{C_1} f$, this proves the claim.

$\blacksquare$

Among other things, the principle of deformation of paths is useful for integrating over complicated contours. Often, we can just replace the contour with a circle.

$\boxed{\text{Example}}$ Let C be any simple closed contour whose interior contains 0. We show that

$$\int_C \frac{1}{z}\, dz = 2\pi i.$$



Since 0 is interior to C, we can choose $\varepsilon > 0$ so small that $C_\varepsilon(0)$ is contained in the interior of C. The region consisting of C, C_ε and the points in between doesn't contain 0, so $\frac{1}{z}$ is analytic there. By deformation of paths

$$\int_C \frac{1}{z}\, dz = \int_{C_\varepsilon} \frac{1}{z}\, dz = \int_0^{2\pi} \frac{1}{\varepsilon e^{it}} \cdot \varepsilon i e^{it}\, dt$$

$$= \int_0^{2\pi} i\, dt = 2\pi i.$$

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More generally, the generalized Cauchy-Goursat theorem and its corollary provide a technique for integrating functions over contours whose interior contains singularities of that function. The idea is to introduce small circles around the singular points, and apply the theorem. It is usually easy to integrate over a circle.

We will use this technique to prove the Cauchy Integral formula and the residue theorem. Exciting!

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Cauchy's Integral Formula

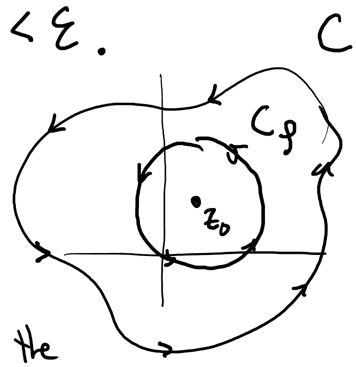
Let C be a simple closed contour. The Cauchy Integral formula is a remarkable theorem. It asserts that, if a function is analytic inside and on C, then its values interior to C are totally determined by its values on C.

Theorem (Cauchy Integral Formula) Let C be a simple closed positively oriented contour. If f is analytic at all points on and interior to C and $z_0 \in \text{Int } C$, then

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0}\, dz .$$

Proof. The idea is to show that, for all $\varepsilon > 0$

$$\left| \int_C \frac{f(z)}{z - z_0}\, dz - f(z_0) 2\pi i \right| < \varepsilon .$$



Let $\varepsilon > 0$. Let $\rho > 0$ so small that $C_\rho(z_0)$ (the circle of radius ρ centered at z_0) is interior to the curve C. Assume C_ρ has positive orientation. Note that $\frac{f(z)}{z - z_0}$ is analytic on the region consisting of C, C_ρ, and the points in between, so by deformation of paths

$$\int_C \frac{f(z)}{z - z_0}\, dz = \int_{C_\rho} \frac{f(z)}{z - z_0}\, dz .$$

Since f is continuous at z_0, choose $\delta > 0$ such that

$$|z - z_0| < \delta \quad \text{implies} \quad |f(z) - f(z_0)| < \frac{\varepsilon}{2\pi} .$$

Shrink ρ so that $\rho < \bar{\rho}$. Then every point $z \in C_\rho(z_0)$ satisfies $|f(z) - f(z_0)| < \frac{\varepsilon}{2\pi}$. Then

$$\left| \int_C \frac{f(z)}{z - z_0}\, dz - f(z_0) 2\pi i \right| = \left| \int_{C_\rho} \frac{f(z)}{z - z_0}\, dz - f(z_0) \int_{C_\rho} \frac{1}{z - z_0}\, dz \right|$$

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$$= \left| \int_{C_\rho} \frac{f(z)}{z - z_0}\, dz - f(z_0) \int_{C_\rho} \right|$$

$$= \left| \int_{C_\rho} \frac{f(z) - f(z_0)}{z - z_0}\, dz \right|$$

Triangle Ineq

$$\leq \max_{z \in C_\rho} \left| \frac{f(z) - f(z_0)}{z - z_0} \right| \cdot 2\pi\rho$$

$$< \frac{\varepsilon}{2\pi\rho} \cdot 2\pi\rho = \varepsilon.$$

▨

Among other things, the Cauchy Integral formula is useful for computing integrals.

$\boxed{\text{Example}}$

(1) $\int_C \frac{\cos z}{z(z^2+2)}\, dz$, C: the positively oriented unit circle

Consider $f(z) = \frac{\cos z}{z^2+2}$. Then f is analytic on and interior to C. By Cauchy integral formula

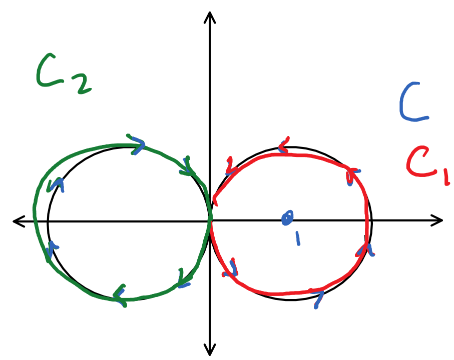
$$\int_C \frac{\cos z}{z(z^2+2)}\, dz = \int_C \frac{f(z)}{z - 0}\, dz = 2\pi i \cdot f(0) = \pi i.$$

(2) $\displaystyle\int_C \frac{e^{z^2}}{z-1}\,dz$, C: the positively oriented circle $|z|=2$.

Consider $f(z) = e^{z^2}$. Then $f(z)$ is entire and hence analytic on and interior to C. Also, 1 is interior to C so

$$\int_C \frac{e^{z^2}}{z-1}\,dz = 2\pi i\, f(1) = 2\pi i e.$$

(3) $\displaystyle\int_C \frac{z^2+1}{z^2-1}\,dz$, C:



The contour C is not simple, but it can be decomposed as a sum of simple closed contours:

$$C = C_1 + C_2.$$

So

$$\int_C \frac{z^2+1}{z^2-1}\,dz = \int_{C_1} \frac{z^2+1}{z^2-1}\,dz + \int_{C_2} \frac{z^2+1}{z^2-1}\,dz.$$

Consider C_1: consider $f(z) = \dfrac{z^2+1}{z+1}$. Then $f(z)$ is analytic inside and on C_1. Also 1 is interior to C_1. By Cauchy's formula,

$$\int_{C_1} \frac{z^2+1}{z^2-1}\,dz = \int_{C_1} \frac{f(z)}{z-1}\,dz = 2\pi i\, f(1) = 2\pi i.$$

For C_2: consider $g(z) = \dfrac{z^2+1}{z-1}$. Then g is analytic inside and on C_2. Also, -1 is interior to C_2. Hence,

$$\int_{C_2} \frac{z^2+1}{z^2-1}\,dz = -\int_{-C_2} \frac{g(z)}{z-(-1)}\,dz = -2\pi i\, g(-1) = 2\pi i.$$

Hence $\displaystyle\int_C \frac{z^2+1}{z^2-1}\,dz = 2\pi i + 2\pi i = 4\pi i.$ //

Theorem (Generalized Cauchy Integral Theorem)

Suppose that f is analytic interior to and on a simple closed positively oriented contour C. If z_0 is interior to C, then

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz.$$